

Fluctuations of Entropy Production in Partially Masked Electric Circuits: Theoretical Analysis

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In this work we perform theoretical analysis about a coupled RC circuit with constant driven currents. Starting from stochastic differential equations, where voltages are subject to thermal noises, we derive time-correlation functions, steady-state distributions and transition probabilities of the system. The validity of the fluctuation theorem (FT)[1, 2] is examined for scenarios with complete and incomplete descriptions.

I. THEORETICAL STUDY: THE COMPLETE DESCRIPTION

We consider the coupled RC circuit as shown in Fig. 1. Two RC circuits of resistances and capacitances (R_1, C_1) and (R_2, C_2) , respectively, are coupled through a third capacitance C_c . The two RC circuits are subject to constant driven currents I_1 and I_2 , and the voltage differences across the resistors are denoted as V_1 and V_2 , respectively. The equation of state of this circuit is

$$\hat{\mathbf{M}}\dot{\vec{V}} + \vec{V} - \vec{\xi} = \vec{V}_d, \quad (1)$$

where $\vec{V} \equiv \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$, $\vec{\xi} \equiv \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$, $\vec{V}_d \equiv \begin{pmatrix} I_1 R_1 \\ I_2 R_2 \end{pmatrix}$, and $\hat{\mathbf{M}} \equiv \begin{pmatrix} R_1(C_1 + C_c) & -R_1 C_c \\ -R_2 C_c & R_2(C_2 + C_c) \end{pmatrix}$. The two resistors are thermalized at temperature T , while voltages across the resistors fluctuate due to the Johnson-Nyquist (thermal) noises ξ_1 and ξ_2 . The noises are assumed to be uncorrelated and Gaussian white, and they satisfy the fluctuation-dissipation relation

$$\langle \vec{\xi}(s) \vec{\xi}^T(s') \rangle = \hat{\mathbf{\Gamma}} \delta(s - s'), \quad \hat{\mathbf{\Gamma}} \equiv \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}, \quad (2)$$

with $\Gamma_m \equiv 2R_m k_B T$ for $m = 1, 2$, where k_B is Boltzmann's constant.

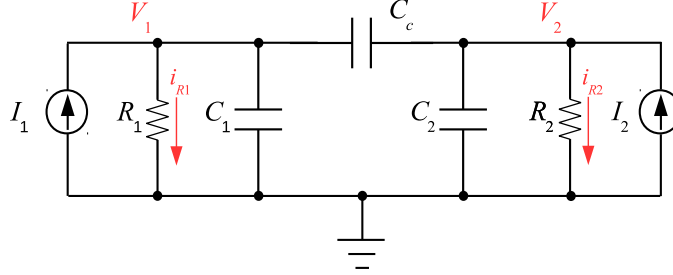


FIG. 1. Illustration of the coupled RC circuit.

Via the change of variables $\vec{V}' \equiv \vec{V} - \vec{V}_d$, the equation of state can be rewritten as $\hat{\mathbf{M}}\dot{\vec{V}}' + \vec{V}' = \vec{\xi}$, which is mathematically identical to that of a non-driven circuit. The solution of the stochastic equation is

$$\vec{V}'(t) = e^{-\hat{\mathbf{M}}^{-1}t} \vec{V}'(0) + \int_{s=0}^t ds \hat{\mathbf{M}}^{-1} e^{-\hat{\mathbf{M}}^{-1}(t-s)} \vec{\xi}(s). \quad (3)$$

In our work we only focus on the steady-state condition, where the first term in Eq. 3 damps out.

Using Eq. 2 and the fact that $\hat{\mathbf{\Gamma}}^{-1}\hat{\mathbf{M}}$ is symmetric, time-correlation functions between voltage signals can be derived as

$$\begin{aligned} & \langle \vec{V}'(t') \vec{V}'^T(t + t') \rangle \\ &= \int_0^{t'} ds \hat{\mathbf{M}}^{-1} e^{-\hat{\mathbf{M}}^{-1}(t'-s)} \hat{\mathbf{\Gamma}} e^{-\hat{\mathbf{M}}^{-1,T}(t+t'-s)} \hat{\mathbf{M}}^{-1,T} \\ &= \int_0^\infty ds \hat{\mathbf{M}}^{-2} e^{-\hat{\mathbf{M}}^{-1}(2t'-2s+t)} \hat{\mathbf{\Gamma}} \xrightarrow[t' \rightarrow \infty]{} \frac{1}{2} \hat{\mathbf{M}}^{-1} e^{-\hat{\mathbf{M}}^{-1}t} \hat{\mathbf{\Gamma}}, \end{aligned} \quad (4)$$

and variances and covariances are just their special cases when $t \rightarrow 0$. Using the method of diagonalization, we then derive

$$\begin{aligned}\langle V_1'(t')V_1'(t'+t) \rangle &= \frac{A_1 e_1/\lambda_1 + A_2 e_2/\lambda_2}{2(\lambda_2 - \lambda_1)^2} \\ \langle V_1'(t')V_2'(t'+t) \rangle &= \langle V_2'(t')V_1'(t'+t) \rangle = \frac{A_1 e_1(\lambda_1 - M_{11})/\lambda_1 + A_2 e_2(\lambda_2 - M_{11})/\lambda_2}{2M_{12}(\lambda_2 - \lambda_1)^2} \\ \langle V_2'(t')V_2'(t'+t) \rangle &= \frac{A_1 e_1(\lambda_1 - M_{11})^2/\lambda_1 + A_2 e_2(\lambda_2 - M_{11})^2/\lambda_2}{2M_{12}^2(\lambda_2 - \lambda_1)^2},\end{aligned}\quad (5)$$

where λ_1 and λ_2 are the eigenvalues of $\hat{\mathbf{M}}$ (λ_1 is assigned as the larger one), $e_m \equiv e^{-t/\lambda_m}$, $A_1 \equiv \Gamma_1(\lambda_2 - M_{11})^2 + \Gamma_2 M_{12}^2$, $A_2 \equiv \Gamma_1(\lambda_1 - M_{11})^2 + \Gamma_2 M_{12}^2$, and M_{mn} are the (m, n) element of the matrix $\hat{\mathbf{M}}$. Moreover, the correlations between \vec{V} and $\vec{\xi}$ can be derived as $\langle \vec{V}'(t')\vec{\xi}^T(t'+t) \rangle = 0$ (no causality) and $\langle \vec{\xi}(t')\vec{V}'^T(t'+t) \rangle = 2\langle \vec{V}'(t')\vec{V}'^T(t+t') \rangle$.

The Fokker-Planck equation of the full circuit can be shown to be

$$\partial P(\vec{V}, t)/\partial t = \nabla \cdot [\hat{\mathbf{M}}^{-1}\vec{V}'P(\vec{V}, t)] + \frac{1}{2}\nabla \cdot \hat{\mathbf{M}}^{-1}\hat{\mathbf{\Gamma}}(\hat{\mathbf{M}}^{-1})^T \nabla P(\vec{V}, t), \quad (6)$$

and the steady-state distribution of \vec{V} is

$$P_{ss}(\vec{V}) \sim \exp(-\vec{V}'^T \hat{\mathbf{\Gamma}}^{-1} \hat{\mathbf{M}} \vec{V}'). \quad (7)$$

For convenience we use the symbol “ \sim ” to denote that the equality holds up to some normalizing constant that remains invariant in the time-reversal process. Note that in the non-driven case the expression reduces to the Boltzmann factor of the stored energy in capacitors.

The transition probability of the complete description, under an infinitesimal change in time, can be derived as

$$P_F(\vec{V}(t+dt)|\vec{V}(t)) \sim e^{-dt \vec{\xi}^T \hat{\mathbf{\Gamma}}^{-1} \vec{\xi}/2} = \exp[-dt (\hat{\mathbf{M}}\dot{\vec{V}} + \vec{V}')^T \hat{\mathbf{\Gamma}}^{-1} (\hat{\mathbf{M}}\dot{\vec{V}} + \vec{V}')/2], \quad (8)$$

where $\dot{\vec{V}} \equiv [\vec{V}(t+dt) - \vec{V}(t)]/dt$, and $\vec{\xi}$ represents the required noises for such transition. In the corresponding time-reversal process,

$$P_R(\vec{V}(t)|\vec{V}(t+dt)) \sim \exp[-dt (-\hat{\mathbf{M}}\dot{\vec{V}} + \vec{V}'')^T \hat{\mathbf{\Gamma}}^{-1} (-\hat{\mathbf{M}}\dot{\vec{V}} + \vec{V}'')/2], \quad (9)$$

as one replaces $\dot{\vec{V}}$ with $-\dot{\vec{V}}$ and \vec{V}' with $\vec{V}'' = \vec{V} + \vec{V}_d$, the latter resulting from inversion of driven currents. Subsequently, the net dissipation can be shown to be

$$\begin{aligned}dS_Q &= k_B \ln \frac{P_F(\vec{V}(t+dt)|\vec{V}(t))}{P_R(\vec{V}(t)|\vec{V}(t+dt))} \\ &= -2k_B dt \vec{V}'^T \hat{\mathbf{\Gamma}}^{-1} \hat{\mathbf{M}} \dot{\vec{V}} + 2k_B dt \vec{V}'^T \hat{\mathbf{\Gamma}}^{-1} \vec{V}_d \\ &= dt \vec{V} \cdot \vec{i}_R / T,\end{aligned}\quad (10)$$

where \vec{i}_R represents the currents through the two resistors. The corresponding change in Shannon entropy is

$$\begin{aligned}dS_{Sh} &= -k_B \ln \frac{P_{ss}(\vec{V}(t+dt))}{P_{ss}(\vec{V}(t))} = -k_B dt \left[\frac{d \ln P_{ss}(\vec{V}(t))}{dt} \right] \\ &= 2k_B dt \vec{V}'^T \hat{\mathbf{\Gamma}}^{-1} \hat{\mathbf{M}} \dot{\vec{V}}\end{aligned}\quad (11)$$

and the total entropy change $dS_{tot} = dS_Q + dS_{Sh}$ is

$$\begin{aligned}dS_{tot} &= -2k_B dt \vec{V}_d^T \hat{\mathbf{\Gamma}}^{-1} \hat{\mathbf{M}} \dot{\vec{V}} + 2k_B dt \vec{V}'^T \hat{\mathbf{\Gamma}}^{-1} \vec{V}_d \\ &= dt (2\vec{V}' \cdot \vec{I}_d - \vec{\xi} \cdot \vec{I}_d + \vec{V}_d \cdot \vec{I}_d) / T.\end{aligned}\quad (12)$$

For our current discussions, \vec{V}_d and \vec{I}_d are constants, and therefore dS_{tot} is a Gaussian random variable, while $\langle dS_{\text{tot}} \rangle = dt \vec{V}_d \cdot \vec{I}_d / T = dt (I_1^2 R_1 + I_2^2 R_2) / T$. One can show that

$$\begin{aligned} \langle dS_{\text{tot}}^2 \rangle - \langle dS_{\text{tot}} \rangle^2 &= (dt)^2 \langle (2\vec{V}' \cdot \vec{I}_d - \vec{\xi} \cdot \vec{I}_d)^2 \rangle / T^2 \\ &= 2k_B dt \vec{V}_d \cdot \vec{I}_d / T = 2k_B \langle dS_{\text{tot}} \rangle. \end{aligned} \quad (13)$$

It is straightforward¹ to demonstrate that FT holds for any Gaussian random variable whose ratio of variance over mean value is $2k_B$. Moreover, with the aid of time-correlation functions, one can also demonstrate the validity of FT over finite-time processes, where $\Delta S_{\text{tot}, \tau} = \int_{t=0}^{\tau} dS_{\text{tot}}$, and $\langle \Delta S_{\text{tot}, \tau} \rangle = \tau \vec{V}_d \cdot \vec{I}_d / T$.

II. REDUCED DESCRIPTION (A): NAIVE DESCRIPTION

In the reduced descriptions, we neglect the signal V_2 intentionally, and we would like to check whether FT can still validate with the knowledge of V_1 only. Note that since the current through R_1 is not measured, the actual dissipation through the resistor is not known, while there exist many methods towards guessing an effective dissipation simply from the time series of V_1 . In this work we adopt two methods. In description (A), we treat the time series of V_1 as that from a virtual single-RC circuit, and compute the current and therefore dissipation directly following the equation of this simplified circuit. And in description (B), an effective dissipation can be derived using the ratio of forward and backward transition probabilities in V_1 over infinitesimal timesteps.

We first derive the steady-state probability distribution in V_1 :

$$P_{\text{ss}}(V_1) = \int dV_2 P_{\text{ss}}(V_1, V_2) \sim \exp\left(-\frac{V_1'^2}{M_{11}^I \Gamma_1}\right) = \exp\left(-\frac{\tilde{C}_1 V_1'^2}{2k_B T}\right), \quad (14)$$

where $\hat{\mathbf{M}}^I \equiv \hat{\mathbf{M}}^{-1}$, and $\tilde{C}_1 \equiv C_1 + \frac{C_2 C_c}{C_2 + C_c}$ is the effective capacitance. Based on Eq. 14, we can develop a naive interpretation (“description (A)”), where the masked circuit is treated as a single-RC circuit with capacitance \tilde{C}_1 and unmodified R_1 and I_1 . Thus R_2 and I_2 are neglected intentionally. This effective single-RC circuit can give the correct steady-state distribution in V_1 . Alternatively, one can regard the time series of V_1 as that from a single-RC circuit, as the effective resistance and capacitance can be derived from its power spectrum, which can be shown to be identical with R_1 and \tilde{C}_1 , respectively.

The total entropy change of this *gedanken* single-RC circuit, during an infinitesimal timestep, is

$$d\tilde{S}_{1\text{tot}}^{(A)} = dt V_1 \tilde{i}_1 / T - k_B \ln \frac{P_{\text{ss}}(V_1(t+dt))}{P_{\text{ss}}(V_1(t))}, \quad (15)$$

where the first term on the RHS represents a “virtual” dissipation, as $\tilde{i}_1 \equiv I_1 - \tilde{C}_1 \dot{V}_1$ is the virtual current going through R_1 in this single-RC circuit. For the case of finite-time difference we have

$$\Delta \tilde{S}_{1\text{tot}, \tau}^{(A)} = \frac{1}{T} \int_0^{\tau} dt (V_1 I_1 - V_1 \tilde{C}_1 \dot{V}_1). \quad (16)$$

Again one finds $\Delta \tilde{S}_{1\text{tot}, \tau}^{(A)}$ to be Gaussian, while $\langle \Delta \tilde{S}_{1\text{tot}, \tau}^{(A)} \rangle = I_1^2 R_1 \tau / T$ is the average virtual dissipation from R_1 . Using time-correlation functions, it is straightforward to derive its variance:

$$\begin{aligned} \langle (\Delta \tilde{S}_{1\text{tot}, \tau}^{(A)})^2 \rangle - \langle \Delta \tilde{S}_{1\text{tot}, \tau}^{(A)} \rangle^2 &= \\ &= \frac{2k_B I_1^2 R_1}{T} \left\{ \tau - \frac{R_1 C_c^2}{C_2 + C_c} + \frac{M_{12} M_{21}}{M_{22}^2 (\lambda_1 - \lambda_2)} \cdot [\lambda_1 (M_{22} + \lambda_2) e^{-\tau/\lambda_1} - \lambda_2 (M_{22} + \lambda_1) e^{-\tau/\lambda_2}] \right\}. \end{aligned} \quad (17)$$

Therefore, FT fails with the adoption of such dissipation function, even at the small- τ limit. Nevertheless, from Eq. 17 one finds that this deviation becomes less prominent at large τ . Moreover, one can also show that the deviation from FT diminishes in the weak-coupling regime, as the deviation in variance from $2k_B \langle \Delta \tilde{S}_{1\text{tot}, \tau}^{(A)} \rangle$ is proportional to C_c^2 .

¹ Consider a Gaussian random variable x of average x_0 and width σ_x . The corresponding symmetry function is $\text{Sym}(x) = \ln[P(x)/P(-x)] = 2x_0 x / \sigma_x^2$. Thus Gaussianity is a sufficient condition of the FT-like behavior. FT is valid if $\sigma_x^2 = 2x_0$ (in dimensionless units; in cases where x means entropy then FT validates if $\sigma_x^2 = 2k_B x_0$), and for the case where the observed slope is not equal to 1, FT can be easily restored with the rescaled variable $x' = (2x_0 / \sigma_x^2) x$.

III. REDUCED DESCRIPTION (B): TRACE-OUT APPROACH

Beside the above reduced description, one can define the effective dissipation function starting from the forward transition probability of V_1 over infinitesimal timesteps. It can be derived by tracing out the degree of freedom in V_2 :

$$\begin{aligned} P_F(V_1(t+dt)|V_1(t)) &= \iint dV_2(t)dV_2(t+dt)P_F(\vec{V}(t+dt)|\vec{V}(t))P_{ss}(\vec{V})/P_{ss}(V_1) \\ &\sim \exp\left\{-\frac{dt}{2(\vec{M}_1^I)^T\hat{\Gamma}\vec{M}_1^I}\left[\dot{V}_1 + \frac{(\vec{M}_1^I)^T\hat{\Gamma}\vec{M}_1^IV_1'}{M_{11}^I\Gamma_1}\right]^2\right\} \\ &\equiv \exp[-dt(\tilde{M}\dot{V}_1 + V_1')^2/(2\tilde{\Gamma}_1)] \end{aligned} \quad (18)$$

to the lowest nonvanishing order, where \vec{M}_1^I is a two-dimensional vector of elements M_{11}^I and M_{12}^I . Note that this transition probability can be compared to that of a single-RC circuit with the aforementioned effective capacitance $\tilde{C}_1 \equiv C_1 + \frac{C_2C_c}{C_2+C_c}$, $\tilde{M} = \tilde{R}_1\tilde{C}_1 = \frac{M_{11}^I\Gamma_1}{(\vec{M}_1^I)^T\hat{\Gamma}\vec{M}_1^I} = (M_{11}^I)^{-1}/(1+\alpha)$, $\tilde{R}_1 = R_1/(1+\alpha)$, and $\tilde{I}_1 = I_1(1+\alpha)$, where $\alpha \equiv \frac{M_{12}M_{21}}{M_{22}^2} = \frac{R_1C_c^2}{R_2(C_2+C_c)^2}$. And the renormalized noise amplitude parameter is

$$\tilde{\Gamma}_1 = (\vec{M}_1^I)^T\hat{\Gamma}\vec{M}_1^I\tilde{M}^2 = \Gamma_1/(1+\alpha), \quad (19)$$

which is smaller than Γ_1 . Since $\exp[-\tilde{C}_1V_1'^2/(2k_B T)] = \exp(-\tilde{M}V_1'^2/\tilde{\Gamma}_1)$, this effective single-RC circuit also gives the correct steady-state probability distribution in V_1 .

In this effective single-RC circuit, the reversed transition probability is derived simply by replacing \dot{V}_1 with $-\dot{V}_1$ and V_1' with $V_1 + V_{1d}$ in Eq. 18. The net dissipation and total entropy change are

$$d\tilde{S}_{1Q}^{(B)} = k_B \ln \frac{P_F(V_1(t+dt)|V_1(t))}{P_R(V_1(t)|V_1(t+dt))} = 2k_B dt[V_1(V_{1d} - \tilde{M}\dot{V}_1)/\tilde{\Gamma}_1] \quad \text{and} \quad (20)$$

$$d\tilde{S}_{1\text{tot}}^{(B)} = d\tilde{S}_{1Q}^{(B)} - k_B \ln \frac{P_{ss}(V_1(t+dt))}{P_{ss}(V_1(t))} = 2k_B dtV_{1d}(V_1 - \tilde{M}\dot{V}_1)/\tilde{\Gamma}_1, \quad (21)$$

respectively. Since $d\tilde{S}_{1\text{tot}}^{(B)}$ is linear in V_1 and \dot{V}_1 , the total entropy change is Gaussian. One can prove that FT is satisfied for $d\tilde{S}_{1\text{tot}}^{(B)}$. However, violation still occurs in finite-time processes, where

$$\langle \Delta\tilde{S}_{1\text{tot},\tau}^{(B)} \rangle = 2k_B\tau V_{1d}^2/\tilde{\Gamma}_1 = I_1^2 R_1 \tau (1+\alpha)/T, \quad (22)$$

and

$$\begin{aligned} \langle (\Delta\tilde{S}_{1\text{tot},\tau}^{(B)})^2 \rangle - \langle \Delta\tilde{S}_{1\text{tot},\tau}^{(B)} \rangle^2 &= \frac{2k_B I_1^2 R_1}{T} \left\{ (1+\alpha)^2 (\tau - M_{11}) + R_1 \tilde{C}_1 \right. \\ &\quad \left. + (1+\alpha)^2 \frac{(\lambda_2 - M_{11})\lambda_1 e^{-\tau/\lambda_1} - (\lambda_1 - M_{11})\lambda_2 e^{-\tau/\lambda_2}}{\lambda_2 - \lambda_1} \right. \\ &\quad \left. - \frac{(R_1 \tilde{C}_1)^2}{\lambda_2 - \lambda_1} \left[\left(\frac{\lambda_2 - M_{11}}{\lambda_1} \right) e^{-\tau/\lambda_1} - \left(\frac{\lambda_1 - M_{11}}{\lambda_2} \right) e^{-\tau/\lambda_2} \right] \right\}. \end{aligned} \quad (23)$$

Note that, on average, the reduced description (B) gives a larger total entropy change than description (A).

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- [1] G. E. Crooks, Phys. Rev. E **60**, 2721 (1999).
 [2] U. Seifert, Rep. Prog. Phys. **75**, 126001 (2012).